

# The Learning Curve and Optimal Investment Under Uncertainty

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## Abstract

In this article we investigate the properties of the optimal investment under uncertainty, irreversibility and learning curve. The main result of our model can be summarized as follows. Because of the effects of the learning, in a context of decreasing returns of capital and incremental investment, a region of increasing returns is endogenously generated and the optimal initial investment is lumpy. Furthermore, as the level of installed capacity moves towards zero, the value of waiting due to uncertainty becomes negligible, and the optimal strategy approximates the traditional NVP rule. Finally, lumpiness in the investment behavior has non-trivial implications for comparative static results. Specifically, factors that delay investment increase its intensity when it eventually occurs.

## 1 Introduction

The learning curve hypothesis states that marginal costs decrease with cumulative production. By producing, the firm exploits a process of learning-by-doing that will guarantee, *ceteris paribus*, higher operating profits in the future. For some industries, like airframes, shipbuilding, nuclear plants, metal and chemical products, there are evidences of declining marginal costs. Although in other sectors is more difficult to obtain reasonable estimates of the firms' marginal cost structure, the learning-by-doing is nonetheless deemed to be a quite common phenomenon in several production processes. Even without the direct support of empirical evidence, the economic intuition suggests that the learning curve might be pervasive in industrial economies.

Majd and Pindyck (1989) show that, under the learning curve hypothesis, the conditions that make the firm willing to produce are less stringent. More precisely, the firm will produce when the marginal revenue is lower than the marginal cost because production, in addition to value of the generated profits, has also a *shadow value* represented by the future reduction in marginal costs. However, in an uncertain world, the effect of the learning curve is weakened because volatility lowers the shadow

value of production and, thus, increases the critical price required for the firm to produce.

In their model, Majd and Pindyck focus only on production choices, assuming a fixed capital stock and a constant rate of production, so that the firm takes the speed of learning as given. What remains as an open issue is the analysis of how the learning curve affects investment behavior. In our model we take on this task by investigating the effects of the learning curve on capital accumulation and optimal investment decisions when the future profits are uncertain. Production choices are not considered because the firm is not given the possibility of temporary shutting down but, contrary to Majd and Pindyck (1989), the capital stock is not taken as fixed and we allow for the possibility of accelerating the learning process by increasing the productive capacity.

Beside the learning curve hypothesis and uncertainty, a third crucial feature of our model is the irreversibility of the investment. Since the seminal paper of Arrow (1968), irreversibility has been widely recognized as a crucial factor influencing the investment choice. If investment is irreversible, the cost of installing new capital are sunk and, combined with uncertainty, this implies that, by investing, the firm exercises an option to wait, to observe the evolution of the stochastic environment and to install additional capacity any time in the future. The well established result of the investment under uncertainty literature shows how the economic conditions that justify the investment are more stringent than the traditional NVP rule because the option of waiting must be incorporated in the investment decision.

The common approach of the investment under uncertainty literature is to focus on discrete investment choices, for instance whether to build a factory or to employ resources to finance a new project. In our model, following Pindyck (1988), we investigate marginal investment decisions. This helps to emphasize the main result of the article: in a model of incremental investment and decreasing marginal returns, the learning curve effect endogenously generates a region of increasing returns and the presence of jumps in the optimal investment behavior. Specifically, the firm will make an initial lumpy investment and will invest marginally in the subsequent units of capital. The possibility to learn induces the firm to start the production with a discrete amount of capital stock to move quickly down along the learning curve. Furthermore, lumpiness in the initial investment choice has surprising implications in terms of comparative statics: events that delay investment increase its intensity when it eventually occurs. For instance, given that uncertainty increases the value of waiting, firms affected both by a significant learning curve and high volatility will be characterized by large lumpy investments. From an aggregate perspective, the growth process of highly volatile sectors will be characterized by occasional and sudden periods of expansion as new firms enter the market.

A surprising result is also found for the combined effect of learning and uncertainty. As the already installed capacity moves towards zero, the value of waiting due to uncertainty becomes negligible and the optimal strategy approximates the traditional

NVP rule. A small capacity implies that the opportunity to learn remains virtually unexploited and, once the expected flow of profits equals the cost of the capital, there is no reason to further delay the investment. Thus, while Majd and Pindyck (1989) found that uncertainty, lowering the shadow value of production and increasing the critical price required for the firm to produce, weakens the effect of learning in the optimal *production* choice, we show how the same is not true, when the level of installed capacity is small, for the optimal *investment* choice.

The article is organized as follows: section 2 formalizes our assumptions and develops the model used for our analysis, section 3 investigates the optimal investment behavior, section 4 presents the results and discusses their implications, section 5 concludes.

## 2 The model

Consider a firm that sells its product at a price determined by the inverse demand function

$$P = X - \varphi q, \quad (1)$$

where  $q$  is the quantity of output produced by the firm in each period and  $X$  can be interpreted as an exogenous demand shock. The exogenous shock  $X$  is stochastic and evolves according to the process

$$\frac{dX}{X} = \alpha dt + \sigma dz, \quad (2)$$

where  $\alpha$  is the expected rate of change of  $X$ , the parameter  $\sigma$  is the standard deviation of this rate of change, and  $dz$  is the increment of a Wiener process. Equation (2) simply says that for given demand shock  $X$  and quantity  $q$  the level of price is fixed, while the future levels of demand are unknown and  $X$  is lognormally distributed with a variance that increases with the time horizon. Thus, the firm faces uncertainty about its future selling price and expected stream of profits. Each unit of the capital stock  $K$  produces one unit of output,  $q = K$ , and can be installed by paying a sunk cost equal to  $w$ . Furthermore, capital stock doesn't depreciate and investment is fully irreversible, so that  $K(t + dt) \geq K(t)$ . This implies that, by installing additional capacity, the firm gives up the opportunity to wait and to observe the evolution of the stochastic demand.

The distinctive feature of our model is the presence of a learning curve. Starting from an initial level  $\bar{c}$ , the marginal cost  $C(Q)$  declines with the cumulative output  $Q$ , until a lower bound  $\underline{c}$ , achieved when  $Q = Qm$ . The marginal cost function is

$$c(Q) = \begin{cases} \bar{c}e^{-\gamma Q} & \text{if } Q < Qm \\ \underline{c} & \text{if } Q \geq Qm. \end{cases} \quad (3)$$

In order to ensure continuity of the marginal cost function we impose  $\gamma = -\frac{1}{Qm} \log(\frac{\underline{c}}{\bar{c}})$ , so that the equality  $c(Qm) = \bar{c}e^{-\gamma Qm} = \underline{c}$  is always satisfied.

At period  $t = 0$ , the firm decides the level of capacity to be installed. In the following periods it can choose to increase the capital stock on the basis of the evolution of the stochastic variable  $X$ . Given that there is no time to build, once decided to invest, the firm can instantaneously install the desired amount of capacity. Finally, I assume that the firm doesn't have the opportunity of suspending the production process. Once the first unit of capacity has been installed, the firm is bounded to produce for every level of selling price  $P$ , or, in other words, whatever the realization of the stochastic variable  $X$  will be. We can assume that the suspension of the production process is costly and these costs are infinite or extremely high, and the only reasonable alternative to production is closing down the plants. Possible implications of this assumption are discussed in the conclusion.

Because of the presence of uncertainty, the total value of the firm  $\Omega(X; Q; K)$  will be given by the value, in terms of discounted stream of profits, of the installed capacity  $V(X; Q; K)$  plus a growth option  $F(X; Q; K)$ , that is the option value to invest in the subsequent units of capital<sup>1</sup>. Hence,

$$\Omega(X; Q; K) = V(X; Q; K) + F(X; Q; K) \quad (4)$$

Given that our model the different units of capital are installed sequentially, the total value of the firm can be also indicated as

$$\Omega(X; Q; K) = \sum_{k=0}^K \Delta V(X; Q; k) + \sum_{k=K}^{\infty} \Delta F(X; Q; k), \quad (5)$$

where  $\Delta V(X; Q; k)$  is value of the  $k + 1$ st unit of capacity, or the present value of expected flow of profits generated by the incremental unit  $k + 1$ , and  $\Delta F(X; Q; k)$  is the value of the option to buy the unit  $k + 1$  at any time in the future minus its cost. If the incremental unit becomes infinitesimally small, we can write equation (5) as

$$\Omega = \int_0^K \Delta V(X; Q; k) dk + \int_K^{\infty} \Delta F(X; Q; k) dk. \quad (6)$$

The optimal strategy for the firm is to choose that level of capital stock  $K$  (i.e. that level of per period rate of production) that maximizes the total value. The maximization problem can be expressed as

$$\underset{K}{Max} (\Omega - wK).$$

The first order condition is  $\Delta V(X; Q; K) - \Delta F(X; Q; K) - w = 0$  and the optimal level of capital  $K^*$  will satisfy  $\Delta V(X; Q; K^*) = \Delta F(X; Q; K^*) + w$ . This condition simply says that the marginal benefit of the incremental unit of capacity should equal the marginal cost represented by the cost of the additional unit plus the opportunity cost of irreversibly exercising the option.

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<sup>1</sup>The term  $F(X, Q, K)$  can also be interpreted as the present value of expected flow of profits generated by potential future investments minus the present value of the cost of that investment.

### 3 Marginal investment

#### 3.1 Learning effect ( $Q < Q_m$ )

In our model the learning process is not infinite because when the marginal cost reaches the lower bound  $\underline{c}$  cannot be further decreased. For this reason the analysis of the decision to invest can be split in two parts. In the first part, we analyze the value of the additional unit when the learning effect is present, that is when the level of cumulative production is below the boundary  $Q_m$  and  $C(Q) = \bar{c}e^{-\gamma Q} > \underline{c}$ . In the second part we focus on the region where the cumulative production is equal or already beyond the boundary  $Q_m$  and the marginal cost is constant at  $\underline{c}$ .

When  $Q < Q_m$ , the per period profit flow generated by the capital stock is

$$\pi(X; Q; K) = (X - \varphi K - \bar{c}e^{-\gamma Q})K.$$

If the firm invests when the learning process is still to be completed, it will obtain from the additional unit of capacity a per period profit flow given by

$$\Delta\pi(X; Q; K) = \begin{cases} X - 2\varphi K - \bar{c}e^{-\gamma Q} & \text{if } Q < Q_m \\ X - 2\varphi K - \underline{c} & \text{if } Q \geq Q_m. \end{cases}$$

Indeed, in the early stages of production, the marginal cost is declining. But once  $Q$  equals  $Q_m$ , the learning process is complete and the marginal cost remains constant at  $\underline{c}$ . Note that, given the impossibility of temporary suspension of the production, this profit flow can also be negative. The value of the additional unit, given the current capacity level  $K$ , is simply the present value of the expected profits:

$$\Delta V(X; Q; K) = \int_0^\infty \int_0^\infty \Delta\pi(X; Q; K) f(X, t) dX e^{-\rho t} dt,$$

where  $f(X, t)$  is the density function for  $X$  at time  $t$ , and  $\rho$  is the exogenously specified discount rate, or

$$\Delta V(X; Q; K) = E \int_0^\infty \Delta\pi(X; Q; K) e^{-\rho t} dt. \quad (7)$$

The integral in (7) can be directly evaluated:

$$\Delta V(X; Q; K) = \frac{1}{\rho - \alpha} X - \frac{1}{\rho} 2\varphi K - \frac{\bar{c}e^{-\gamma Q}}{\gamma K + \rho} (1 - e^{-(\gamma K + \rho)T}) - \frac{\underline{c}}{\rho} e^{-\rho T}, \quad (8)$$

where  $T = \frac{Q_m - Q}{K}$  is the number of periods needed to reach the marginal cost lower bound  $\underline{c}$ . As it should be clear, additional capital stock boosts the rate of production and speeds up the learning process.

The the option value of investing in an additional unit of capacity  $\Delta F(X; Q; K)$  is derived through standard dynamic programming. As shown in the Appendix,  $\Delta F(X; Q; K)$  is the solution of

$$\frac{1}{2}\sigma^2 X^2 \Delta F_{XX} + \alpha X \Delta F_X + K \Delta F_Q - \rho \Delta F = 0. \quad (9)$$

The above partial differential equation must be solved under the following boundary conditions

$$\Delta F(0; Q; K) = 0 \quad (10)$$

$$\Delta V(X^*; Q; K) = \Delta F(X^*; Q; K) + w \quad (11)$$

$$\Delta V_X(X^*; Q; K) = \Delta F_X(X^*; Q; K) \quad (12)$$

Using equation (8), conditions (11) and (12) can be rewritten as

$$\begin{aligned} \frac{1}{\rho - \alpha} X - \frac{1}{\rho} 2\varphi K - \frac{\bar{c}e^{-\gamma Q}}{\gamma K + \rho} (1 - e^{-(\gamma K + \rho)T}) - \frac{\underline{c}}{\rho} e^{-\rho T} &= \Delta F(X^*, Q, K) + w \\ \frac{1}{\rho - \alpha} &= \Delta F_X(X^*; Q; K). \end{aligned}$$

Equation (10) says that if the level of the stochastic variable  $X$  is zero the option to invest in the future periods has no value. Equations (11) and (12) are the value matching and smooth pasting conditions at the boundary level  $X^*$  that divides the investment and non investment regions. The partial differential equation (9) has no analytical solution and we solve it numerically using the finite difference method<sup>2</sup>.

### 3.2 The boundary $Qm$

When the cumulative production is already equal or beyond  $Qm$ , the learning process is complete: increasing the cumulative production does not decrease the marginal cost and, thus, doesn't affect the value of the additional unit of capacity. The per period profit flow depends only on the fluctuations of  $X$ , and is given by

$$\Delta \pi(X; K) = X - \underline{c} - 2\varphi K.$$

The value of the additional unit of capacity is, as before, the present value of the expected profits:

$$\Delta \bar{V}(X; K) = \int_0^\infty \int_0^\infty (X - \underline{c} - 2\varphi K) f(X, t) e^{-\rho t} dt,$$

where  $f(X, t)$  is the density function for  $X$  at time  $t$ , or

$$\Delta \bar{V}(X; K) = E \int_0^\infty (X - \underline{c} - 2\varphi K) e^{-\rho t} dt.$$

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<sup>2</sup>Details of the numerical procedure are described in the Appendix.

The above integral can be directly evaluated:

$$\Delta \bar{V}(X; K) = \frac{1}{\rho - \alpha} X - \frac{1}{\rho} (\underline{c} + 2\varphi K). \quad (13)$$

Using dynamic programming, as in the previous section, it is possible to show that the option value to invest in the additional unit of capacity is the solution of the following differential equation:

$$\frac{1}{2} \sigma^2 X^2 \Delta F_{XX} + \alpha \Delta F_X - \rho \Delta F = 0. \quad (14)$$

Note that in this case  $\Delta F$  is defined by an ordinary differential equation, while in the above section was the solution of a partial differential equation. The difference is due to the fact that, when the cumulative production has already reached the boundary  $Qm$ , the option to invest is not affected by changes in the cumulative production and depends only on the fluctuations of the stochastic variable  $X$ . The general solution to equation (14) is

$$\Delta F = AX^{\beta_1} + BX^{\beta_2},$$

where

$$\beta_1 = \frac{1}{2} - \alpha/\sigma^2 + \sqrt{(\alpha/\sigma^2 - \frac{1}{2})^2 + \frac{2\rho}{\sigma^2}} > 1$$

$$\beta_2 = \frac{1}{2} - \alpha/\sigma^2 - \sqrt{(\alpha/\sigma^2 - \frac{1}{2})^2 + \frac{2\rho}{\sigma^2}} < 0.$$

At  $X = 0$  the option to invest has no value, so that

$$\Delta F(0, K) = 0.$$

which implies that  $B = 0$  and

$$\Delta F = AX^{\beta_1}. \quad (15)$$

At the boundary defining the investment and non investment regions it must hold that

$$\Delta V(X^*; K) = \Delta F(X^*, K) + w \quad (16)$$

$$\Delta V_X(X^*; K) = \Delta F_X(X^*, K) \quad (17)$$

Substituting equations (13) and (15), the above conditions can be rewritten as

$$\frac{1}{\rho - \alpha} X^* - \frac{1}{\rho} (\underline{c} + 2\varphi K) = AX^{*\beta_1} + w$$

$$\frac{1}{\rho - \alpha} = \beta_1 AX^{*\beta_1 - 1}.$$

Conditions (16) and (17) provide us with two equations to find the two unknowns  $X^*$  and  $A$ :

$$X^* = \frac{\beta_1(\rho - \alpha)}{\beta_1 - 1} \left( \frac{\underline{c}}{\rho} + \frac{2\varphi K}{\rho} + w \right) \quad (18)$$

$$A = \frac{1}{\beta_1(\rho - \alpha)} X^{*(1-\beta_1)} \quad (19)$$

## 4 Investment decisions

### 4.1 Optimal investment boundary

In this section we find the value of the additional unit of capacity and define the optimal production rule. As explained in section 3, the solution to the firm's investment choice involves the determination of three elements: the value of the additional unit  $\Delta V(X; Q; K)$ , the value of the option to invest  $\Delta F(X; Q; K)$  and the investment threshold  $X^*$ . When the learning effect is present, the term  $\Delta V(X; Q; K)$  is given by equation (8),  $\Delta F(X; Q; K)$  is the solution of the partial differential equation (9) and  $X^*$  is determined by the value matching and smooth pasting conditions defined by equations (11) and (12).

The partial differential equation (9) has no analytical solution and we solve it numerically using the finite difference method. The parameters values are set as follows: initial marginal cost  $\bar{c} = 40$ , final marginal cost  $\underline{c} = 10$ , volatility  $\sigma = 0.2$ . The level of cumulative production at which the learning process is complete is  $Q_m = 100$ , the exogenously specified discount rate  $\rho = 0.1$ , and  $\alpha = 0.07$ . The parameter  $\varphi$  is set equal to 0.8,  $w = 50$  and the level of the already installed capacity is  $K = 1$ .

Table 1 shows the results of the numerical simulation. Each column compares for a given cumulative production, i.e. for a given level of marginal cost, the net value  $\Delta V - w$  of the installed unit of capacity and the option value to invest  $\Delta F$  for different levels of  $X$ . The starred entries represent the value of the additional unit at the threshold level  $X^*$  that divides the investment and non investment regions. Below  $X^*$  the firm doesn't invest, and the value of the additional unit is given by the option  $\Delta F$ . When  $X$  is above the threshold  $X^*$ , the firm invests and the value of the additional unit is given by the present value of the expected stream profits from the additional unit of capital  $\Delta V$ , minus the sunk installation cost  $w$ . For example, when the cumulative production is  $Q = 20$ , the threshold value is  $X^* = 42.58$ . For this level of  $X$ , the net value of the installed unit and the option to invest are the same (because of the value matching condition), so that  $\Delta V - w = \Delta F = 1086.9$ . For  $X \geq X^*$ , the firm will invest, and the numerical values above the boundary are given by  $\Delta V - w$ . If  $X < X^*$ , the firm will wait, so that below the boundary the numerical values are given by  $\Delta F$ . When the learning process is complete, i.e.  $Q = 100$  and  $C(Q) = \underline{c} = 10$ , the boundary between investment and non investment regions is given  $X^* = 20.95$ , and for that value  $\Delta V - w = \Delta F = 527.34$ . For  $X$  equal or above 20.95 the firm will invest, and the other way around.

A more intuitive picture of the behavior of the free boundary  $X^*$  is given by Fig.1. As it can be seen,  $X^*$  is maximal for a cumulative production equal to zero and it is exponentially decreasing as  $Q$  increases. When the cumulative production is low, the marginal cost is close to its upper bound. It follows that, for a given  $X$ , the value of the additional unit is lower and the firm will need a higher level of price to be induced to exercise the option. For high levels of cumulative production, the marginal cost is

$X$	Cumulative production (current marginal cost)					
	0 (40)	20 (30.12)	40 (22.98)	60 (17.41)	80 (13.20)	100 (10)
53.47	1365*	1450.1	1514.5	1563.2	1598.8	1616.3
50.5	1268.7	1351.1	1415.5	1464.2	1499.8	1517.3
47.86	1184.4	1263.1	1327.5	1376.2	1411.8	1429.3
45.22	1101.5	1175.1	1239.5	1288.2	1323.8	1341.3
42.58	1019.8	1086.9*	1151.5	1200.2	1235.8	1253.3
39.94	939.62	1001.3	1063.5	1112.2	1147.8	1165.3
37.96	880.42	938.18	997.53	1046.2	1081.8	1099.3
35.98	822.07	875.95	931.53	980.2	1015.8	1033.3
34	764.61	814.66	865.5*	914.2	949.8	967.33
32.02	708.08	754.37	801.37	848.2	883.8	901.33
29.71	643.36	685.35	727.96	771.2	806.8	824.33
27.73	588.99	627.38	666.3	705.1	740.8	758.33
26.74	553.34	589.36	625.88	662.25*	696.8	714.33
24.76	509.47	542.59	576.15	609.55	641.8	659.33
23.11	466.42	496.69	527.35	557.84	586.6*	604.33
21.79	432.59	460.62	489.01	517.21	543.71	560.33
20.95	407.58	433.97	460.68	487.19	512.02	527.34*

Table 1: Value of the additional unit and optimal production rule

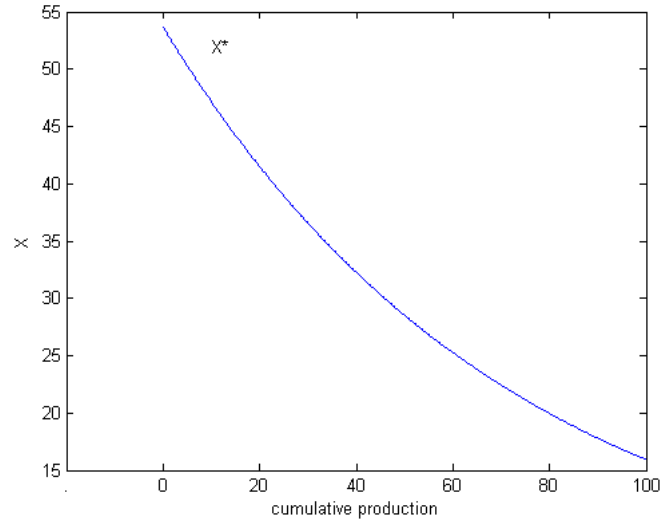


Figure 1: Investment threshold for different levels of cumulative production

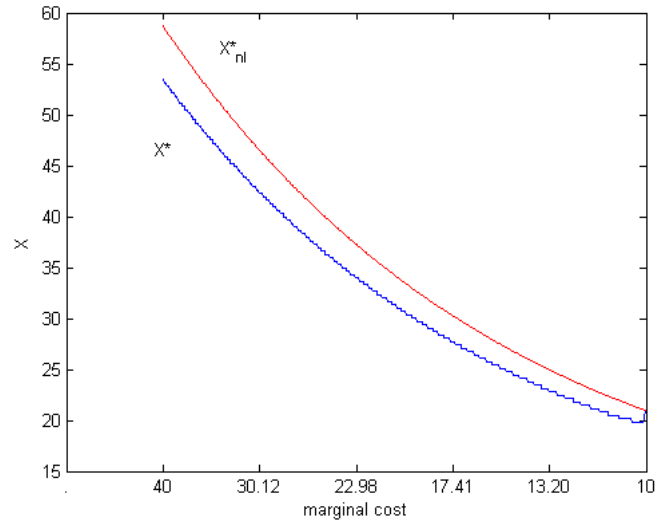


Figure 2: Investment threshold for a learning and a non-learning firm

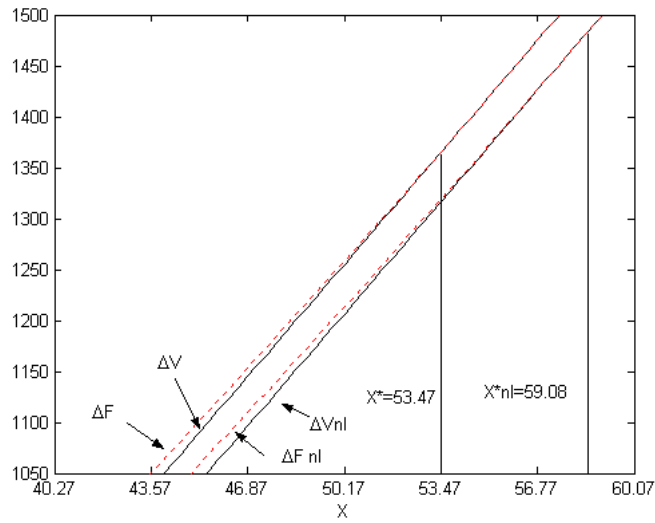


Figure 3: Value and option value of the additional unit for a learning and a non-learning firm

close to its lower bound and, because of the effect of the learning curve, the expected stream of profits will be larger and the threshold value  $X^*$  lower.

In order to grasp the effect of learning, it is instructive to compare the threshold curve of Fig. 1 with the benchmark case of a firm facing constant marginal costs. Recall that in the learning scenario  $X^*$  is determined by eq. (12) and (13). When the marginal cost is constant,  $X_{nl}^*$ , defined as the threshold investment value for a non-learning firm, can be determined in the same way as the case when  $Q \geq Qm$ . Hence,  $X_{nl}^*$  will be given by

$$X_{nl}^* = \frac{\beta_1(\rho - \alpha)}{\beta_1 - 1} \left( \frac{C}{\rho} + \frac{2\varphi K}{\rho} + w \right),$$

where  $C$  is the (constant) marginal cost. Fig. 2 shows that for the learning firm the threshold value is always lower. Indeed, under the learning scenario, the firm knows that the marginal cost will decrease in the future when the cumulative production will increase. It follows that the incentive to invest in the additional unit will be stronger and the threshold value  $X^*$  lower. As it should be clear, at  $Q = 100$ , when the learning process is complete, the two threshold values exactly coincide. Fig. 3 helps to clarify the point. The dashed curves represent the option value to invest unit for the learning firm  $\Delta F$  and the non-learning firm  $\Delta F_{nl}$ , while the solid curves represent the two values of the additional unit  $\Delta V$  and  $\Delta V_{nl}$ . The level of the already installed capacity is equal to one. As it can be seen, for the learning firm value and option value are higher and the threshold level  $X^*$  will result consequently lower ( $X^* = 53.47$  and  $X_{nl}^* = 59.08$  respectively).

## 4.2 Lumpy investments

The results derived in the previous section apply when the firm has strictly positive levels of cumulative production installed. A special case is represented by the situation where  $K = 0$ . Recall that, for  $Q < Qm$ ,  $\Delta V$  and  $\Delta F$  are defined by equations (9) and (10). Evaluating those expressions at  $K = 0$ , and taking into account that with no capacity the cumulative production  $Q$  is equal to zero, yields

$$\Delta V(X, Q, 0) = \frac{1}{(\rho - \alpha)} X - \frac{\bar{c}}{\rho}, \quad (20)$$

$$\frac{1}{2}\sigma^2 X^2 \Delta F_{XX} + \alpha X \Delta F_X - \rho \Delta F = 0. \quad (21)$$

Given that at  $K = 0$  the firm is not producing, the benefits deriving from the learning curve remain unexploited and the term  $K \Delta F_Q$  in the partial differential equation (9) cancels out. It follows that the option value to invest in additional capacity  $\Delta F$  is the solution of the ordinary differential equation (21), with analytical solution

$K$	0	0.5	1	1.5	2	2.5	2.9	3	3.5
$X^*$	56.81	51.49	53.47	54.79	55.45	56.11	56.77	57.1	57.76

Table 2: Investment threshold for different levels of capital

$\Delta F(X, Q, 0) = BX^{\beta_1}$ , where  $\beta_1$  is the positive root of the fundamental quadratic. The equation must be solved under the following boundary conditions.

$$\Delta F(0, Q, 0) = 0 \quad (22)$$

$$\Delta V(X_0^*; K) = \Delta F(X^*, K) + w \quad (23)$$

$$\Delta V_X(X_0^*; K) = \Delta F_X(X^*, K). \quad (24)$$

where  $X_0^*$  is that value of the stochastic variable  $X$  which justifies the initial investment and the start of the production. Equations (23) and (24) are the value matching and smooth pasting conditions. Using equation (20) and the solution of the ordinary differential equation, they can be rewritten as

$$\frac{1}{(\rho - \alpha)} X_0^* - \frac{\bar{c}}{\rho} = BX_0^{*\beta_1} + w$$

$$\frac{1}{(\rho - \alpha)} = \beta_1 BX_0^{*\beta_1 - 1},$$

Value matching and smooth pasting conditions can be solved to find the two unknowns  $X_0^*$  and  $B$ :

$$X_0^* = \frac{\beta_1(\rho - \alpha)}{\beta_1 - 1} \left( \frac{\bar{c}}{\rho} + w \right) \quad (25)$$

$$B = \frac{1}{\beta_1(\rho - \alpha)} X_0^{*(1-\beta_1)}. \quad (26)$$

Once the initial threshold  $X_0^*$  is found, we can thoroughly investigate the properties of the investment behavior. In particular, we will show that, because of the effect of learning, a region of increasing returns is endogenously generated and the initial investment is lumpy. Fig. 4 and Table 2, that show the threshold values for different levels of capital when  $C(Q) = 40$  (i.e.  $Q = 0$ ), shed light on this result.

In Fig. 4, the horizontal line represents the trigger value at  $K = 0$  ( $X_0^* = 56.81$ ), i.e. that value of the stochastic variable  $X$  that induces the firm to make initial investment and to start the production. As it should clear from the figure, for an initial region the trigger values are below 56.81 and increasing in  $K$ . This implies that, once  $X$  reaches the level 56.81, the firm doesn't invest marginally but directly installs a "discrete" amount of productive capacity. Specifically, the firm jumps to that level of capacity for which it holds  $X_0^* = X^* = 56.81$ .

Due to the discrete nature of the numerical procedure, to find the initial optimal capacity we assume, without any loss of generality, that the minimum (marginal)

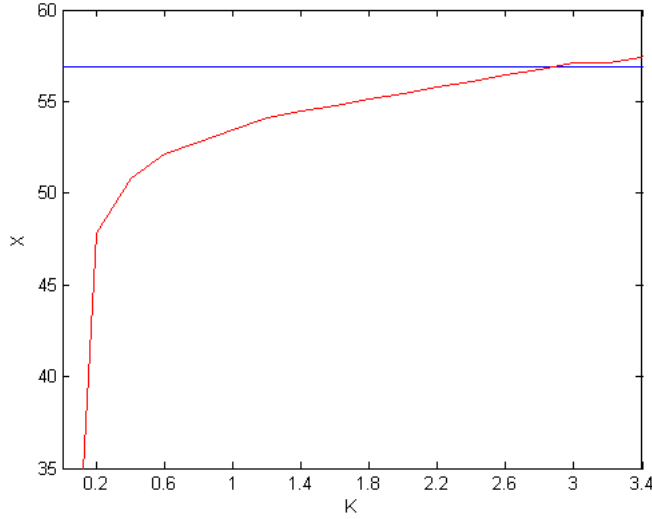


Figure 4:  $X^*$  for different levels of  $K$  and the initial lumpy investment.

investment is equal to  $I = \Delta K = 0.1$ . If, for example, for  $K = 5$  and  $K = 5.1$ , the threshold  $X^*$  is respectively 59.74 and 60.07, and  $X_0^*$  is equal to 59.90, the firm will immediately jump to a capacity of 5.1. For the numerical values in Table 2, this implies that, should  $X$  overtake the initial threshold  $X_0^* = 56.81$ , the firm will instantaneously install a level of capacity  $K = 3$ . Once the initial investment is made, given that  $X^*$  is increasing in  $K$ , the firm will invest marginally in the subsequent units of capital.

In order to have a complete picture of the consequences of the learning effect, it is instructive to see what happens when the level of installed capital is  $K = \varepsilon$ , with  $\varepsilon$  strictly positive and infinitesimally small<sup>3</sup>. In this case, the results of the numerical simulation show that the threshold value is  $X^* = 13.53$  and the value of the additional unit is  $\Delta V(X, Q, 0) = \frac{1}{(\rho - \alpha)} 13.53 - \frac{\bar{c}}{\rho} = 51$ . Recalling that in our parameters set the sunk cost to install an additional unit of capacity is  $w = 50$ , it follows that the optimal investment decision is approximately an NVP rule<sup>4</sup>. The reason is that the incentive to invest due to the learning curve offsets the value of waiting due to uncertainty, so that the value of the option to invest  $\Delta F$  collapses to zero. Fig. 5 shows  $\Delta V$  and  $\Delta F$  as function of  $X$  for  $K = 1$  and  $K = \varepsilon$ . As it should be clear, for  $K = \varepsilon$ , below the threshold value  $X^*(\varepsilon)$ , the option to wait is zero and the optimality condition reduces to the NVP rule  $\Delta V = w^5$ . When the installed capital stock is infinitesimally small,

<sup>3</sup> $K=1e-150$  in our numerical simulation.

<sup>4</sup> $\Delta V$  is not exactly equal to 50 because of discretization error.

<sup>5</sup>The figure is potentially misleading because it seems characterized by a kink at the threshold value  $X^*$ , when the firm install the next unit. The possibility of a kink is by construction excluded because the partial differential equation defining  $\Delta F$ , is solved under the smooth pasting condition.

the firm is producing without exploiting the benefits of learning and, once the value of the additional unit equals its installation cost, there is no reason to further delay the investment. Note a crucial difference with Majd and Pindyck (1989). In their analysis of *production* choices, they found that uncertainty weakens the effect of the learning curve because volatility lowers the shadow value of production and increases the critical price required for the firm to produce. On the contrary, in our analysis of *investment* choices, we show how, when the level of capacity is small, the learning curve effect wipes out the value of waiting deriving from uncertainty.

Another result is worthy of attention. The small step, in terms of productive capacity, from  $K = 0$  to  $K = \varepsilon$  has substantial implications in terms of optimal investment behavior. Indeed, as we have seen above, when  $K = 0$  the investment is lumpy, but in the hypothetical scenario when  $K = \varepsilon$  the firm invests marginally<sup>6</sup>. The reason is that, as shown in Fig.4, once a strictly positive amount of capital is installed,  $X^*$  is increasing in  $K$ . If, for example, the production process has already started (suboptimally) with an infinitesimally small amount of capital installed, once  $X$  overtakes 13.53, the firm will invest in the subsequent marginal unit. On the contrary when  $K = 0$ , if  $X$  goes beyond 13.53, nothing happens because the firm will wait until  $X = 56.81$  to directly install a discrete amount of capital ( $K = 3$ ). It is worth stressing that by investing marginally in the whole region where  $K < 3$ , the firm remains in a suboptimal situation but, in any case, is improving its condition because the capital stock gets closer and closer to the efficient threshold  $K = 3$  or, in other words, farther and farther from the inefficiently low level  $K = \varepsilon$ . Roughly speaking, without waiting to jump directly to  $K = 3$  when  $X$  reaches 56.81, the firm is willing to invest marginally because by adding new capacity the resulting capital stock, even if still suboptimal, will be "less suboptimal".

If a firm invests marginally at  $K = 0$ , it means that the first (marginal) unit is the most valuable. In our context this is not true because an infinitesimally small amount of capital implies an infinite number of periods of production at a marginal cost equal to its upper bound. Thus, the possibility to learn would remain completely unexploited. For this reason, the highest marginal returns will be reached when the level of capital is large enough to guarantee a sufficiently fast learning. In other words, the presence of the learning curve determines an initial region of increasing returns.

Hence, despite our model, by construction, analyzes incremental investment decisions in a context of decreasing returns, we obtain the striking result that a region of increasing returns is endogenously generated, and investment is lumpy for the initial capacity choice. Furthermore, the increasing returns region is characterized by a peculiar nature that deserves a further investigation.

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The apparent kink is due to discretization error.

<sup>6</sup>Recall that  $K = \varepsilon$  is a suboptimal situation. If the firm follows an optimal strategy we will never see it producing with an infinitesimally small amount of capacity (for our baseline parameters set, the minimal capacity size is  $K = 3$ ).

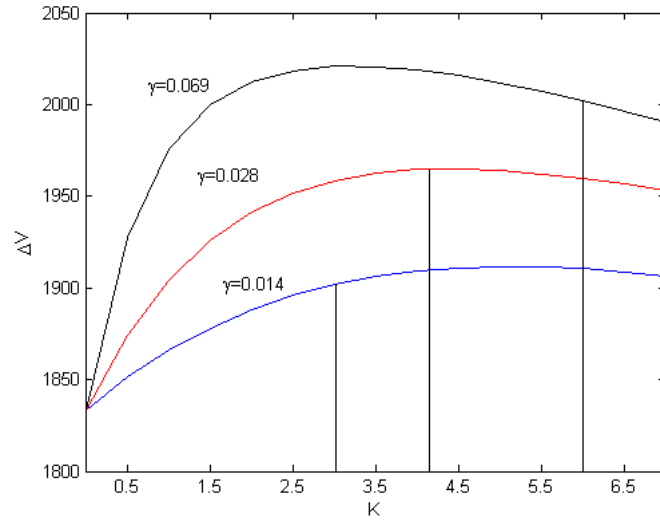


Figure 5: Marginal productivity of capital and initial investment for various levels of  $\gamma$  ( $X = 67$ )

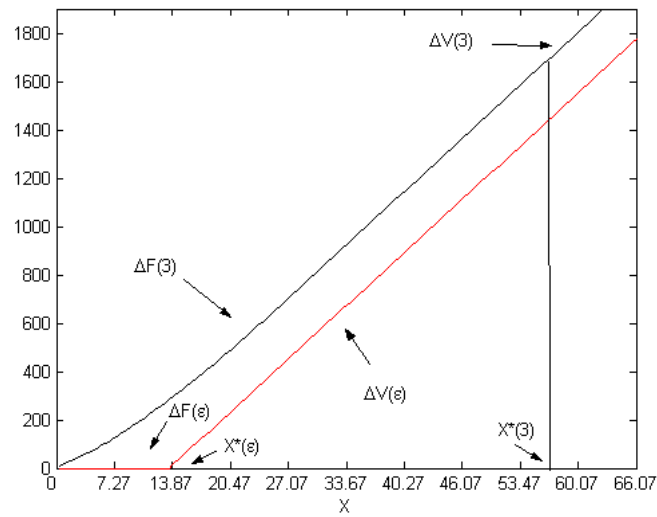


Figure 6:  $\Delta V$  and  $\Delta F$  for  $K = 3$  and  $K = \epsilon$

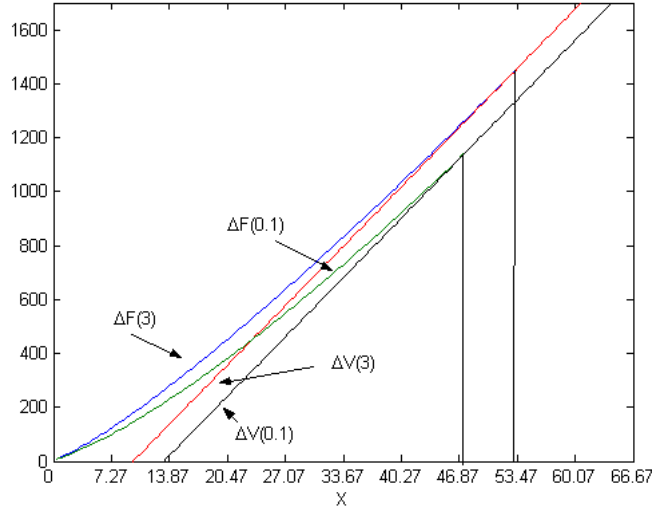


Figure 7:  $\Delta V$  and  $\Delta F$  for  $K = 0.1$  and  $K = 3$  ( $\gamma = 0.069$ )

Fig. 6 shows the  $\Delta V$  curves and the size of the initial investment, (obtained using the condition  $X_0^* = X^*$  and represented by intersection of the vertical lines with the  $x$  axis) for different levels of the learning speed parameter  $\gamma$  and  $X = 67$ . Denoting by  $\bar{K}$  the initial size of the investment, it can be seen, for example, that when  $\gamma = 0.014$  the size of the initial investment is  $\bar{K} = 3$ , when  $\gamma = 0.028$  the initial investment will be  $\bar{K} = 4.2$  while for  $\gamma = 0.069$ ,  $\bar{K} = 6$ . The marginal productivity curves are initially increasing then decreasing. In the regions from  $K = 0$  to the point where the  $\Delta V$  curves achieve their maximum, increasing returns apply. Note the effect of a rise in the learning speed parameter. As  $\gamma$  increases, the value of the additional unit  $\Delta V$  is higher, the increasing returns region is smaller and the size of the initial investment  $\bar{K}$  becomes larger. Indeed, a faster learning makes the additional unit more valuable ( $\Delta V$  is higher) because the marginal cost lower bound is quickly reached and the discounted stream of profits is, consequently, larger. For this reason, the firm is more willing to invest ( $\bar{K}$  is larger)<sup>7</sup> and a lower productive capacity is needed to obtain the highest marginal productivity (i.e. the increasing returns region is smaller).

As shown in Dixit (1995), when marginal returns are increasing, the optimal behavior is characterized by lumpy investments. The firm will jump the increasing returns region and will install an amount of capacity such that the average product per unit of capital is maximized. Typically, for that level of capacity marginal returns of capital are *decreasing*. On the contrary, as it should be clear from Fig. 6, for  $\gamma$  equal 0.014 and 0.028 the predicted initial amount of capital lies in the region of

<sup>7</sup>The effect of  $\gamma$  on the size of the initial investment is more complicated and is extensively analyzed in section 4.3.

*increasing* returns. Hence, following the Dixit's analysis, the "optimal" behavior of the firm predicted by our model is actually suboptimal because the firm install a level of capacity that doesn't maximize the average productivity, or, in other words, the increasing returns region is not fully exploited.

In order to gain insights about this non-standard investment behavior, it should be noticed an apparently counterintuitive result. Once a strictly positive level of capacity is installed, the threshold  $X^*$  is increasing in  $K$  even in the region of increasing returns. In other words, in sharp contrast with Dixit (1995), as  $K$  rises,  $X^*$  increases even if the additional unit is more valuable. Fig. 7, that shows  $\Delta V$  and  $\Delta F$  for  $\gamma = 0.069$  and  $K$  equal to 0.1 and 3, helps to clarify the point. As it can be seen, when the capital stock increases from  $K = 0.1$  to  $K = 3$ , also the threshold value  $X^*$  increases, despite the fact that the value of the marginal unit at  $K = 3$  is larger. This surprising result is a consequence of the learning effect. Recall that the incentive deriving from learning is maximal for  $K = 0$ , where the learning possibilities are unexploited, and decreasing in  $K$ . As  $K$  increases, the learning process becomes faster and the incentive to further accelerate the reduction of marginal cost by adding new capacity becomes weaker. It follows that, if the productive capacity increases, the option value to invest  $\Delta F$  becomes relatively more important with respect to  $\Delta V$ , pushing up the threshold value  $X^*$  even in the region of increasing returns. In other words as  $K$  rises, both  $\Delta V$  and  $\Delta F$  increase, but the growth of the option to wait  $\Delta F$  is proportionally larger, so that the investment trigger  $X^*$  will result higher.

To summarize, the learning curve generates an increasing returns region. In this region, as the capital stock increases, the rise in the option to invest  $\Delta F$  is larger than that of  $\Delta V$ , because the value of waiting due to uncertainty becomes relatively more important than the incentive to invest due to learning. For this reason, even though the value of the additional unit is larger, the investment trigger  $X^*$  is increasing in  $K$ , and this implies that for the initial investment the firm will install a level of capacity that doesn't maximize the average product per unit of capital.

### 4.3 Comparative statics

In this section we perform a comparative static analysis using as baseline set of parameters values the same as section 4.1. As we will see, the presence of jumps in the investment decisions gives rise to non-standard outcomes for what concerns the characteristics of the initial investment.

A well established result in real option analysis states that uncertainty increases the value of waiting, thus enlarging the inaction region. When the volatility is higher, the option of waiting and having the possibility to observe the evolution of the stochastic variable of interest becomes more valuable. In general, an early action in a highly volatile environment is riskier, while waiting allow for the possibility of becoming operative in the "good" states of the world. Our model is not an exception to this standard result. As shown in Table 3, when the volatility  $\sigma$  rises, the threshold  $X_0^*$

$\sigma$	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
$X_0^*$	45.80	48.13	51.84	56.81	62.90	70.08	79.30	87.56
$\bar{K}$	0.8	1.3	2	3	4.4	5.9	7.4	8.7

Table 3: The effects of volatility on the initial investment threshold and size of the jump

$w$	10	30	50	70	90	110	170	230
$X_0^*$	51.75	54.28	56.81	59.33	61.85	64.38	71.95	79.52
$\bar{K}$	2.7	2.8	3	3.2	3.3	3.4	3.6	3.7

Table 4: The effects of sunk installation costs on the initial investment threshold and size of the jump

increases, expanding the non investment region. But, as it can be observed, beside rising the investment trigger, a higher volatility makes the size of the initial investment larger, should this eventually occur. For instance, when  $\sigma = 0.1$ , the initial trigger is  $X_0^* = 48.13$  and the optimal initial capacity is  $\bar{K} = 1.3$ . As the volatility rises to 0.2,  $X_0^*$  increases to 56.81 and the corresponding initial capacity will be  $\bar{K} = 3$ . Thus, uncertainty delays investment (the threshold value is  $X_0^*$  higher) but increases its intensity. In an uncertain scenario, a firm, once the conditions to start the production are met (in terms of the level of the stochastic variable  $X$ ), the intensity of the lumpy investment will be larger the higher is the level of volatility.

The effect of uncertainty on the size of the initial investment deserves also a quantitative discussion. When the firm has a cumulative production equal to zero, the number of periods needed to reach the marginal cost lower bound  $\underline{c}$  is given by  $T = \frac{Qm}{K}$ . Considering our initial parameter values ( $Qm = 100$  and  $\sigma = 0.2$ ), this implies that the firm will need  $T = \frac{Qm}{3} = 33.3$  periods of production before reaching  $\underline{c}$ . For  $\sigma = 0.3$  the optimal initial capacity is equal to  $\bar{K} = 5.9$ , and the learning will be complete in less than 17 periods. For  $\sigma = 0.4$ , and the corresponding optimal starting capacity level  $\bar{K} = 8.7$ , the learning process will last 11.5 periods. For high volatility levels, the firm will enter the market only when the learning process is very fast.

The effects of changes in the sunk installation cost  $w$  are shown in Table 4. As expected, a higher  $w$  increases the initial trigger value  $X_0^*$ . If the sunk installation cost is higher the firm needs a higher level of  $X$  to start the production. Less obvious, *a priori*, is effect on the size of the lumpy investment  $\bar{K}$ . As shown in the table, as  $w$  rises,  $\bar{K}$  becomes larger. The effect is the same as an increase in the volatility level. A rise in the sunk installation cost delays investment ( $X_0^*$  rises) but increases its intensity when it eventually occurs ( $\bar{K}$  also rises). Note that for the already operating firms, whose investment is marginal (because  $X^*$  is increasing in  $K$ ), a rise in the cost of capital unambiguously causes a decline in the investment. But for the

$\alpha$	0.03	0.04	0.05	0.06	0.07	0.08	0.09
$X_0^*$	63	61.12	59.49	58.04	56.81	55.73	54.80
$\bar{X}$	5.1	4.3	3.7	3.2	3	2.7	2.5

Table 5: The effects of growth rate on the initial investment threshold and size of the jump

$\bar{c}$	15	20	30	40	50	60	70
$X_0^*$	25.25	31.56	44.18	56.81	69.43	82.05	94.68
$\bar{K}$	1	1.2	1.9	3	4.4	6.2	8

Table 6: The effects of the marginal cost upper bound on the initial investment threshold and size of the jump

potential start-ups, whose investment is lumpy, the effect is ambiguous. The initial threshold is higher, thus some project might be blocked or temporary suspended, but if and when the economic conditions that justify the initial investment are satisfied, the firm will install a larger capacity.

Table 5 shows the effect of changes in the growth rate of  $\alpha$ . A larger  $\alpha$  decreases the initial trigger value  $X_0^*$ . If the firm expects the price level growing at a faster rate, a lower level of  $X$  is needed to start the production. On the other hand, a higher growth rate decreases the size of the initial investment. It's perhaps more instructive to read the result from the opposite point of view. Starting from a high level of  $\alpha$ , a *decrease* in the growth rate has the same effect of a rise in volatility or in the sunk installation cost: the investment is delayed but it is larger when occurs.

Table 6 shows the effect of changes in the marginal cost upper bound  $\bar{c}$ . In addition to its direct effect, a rise in  $\bar{c}$  has also an indirect effect on the speed of learning. It must be recalled, indeed, that in order to ensure continuity of the marginal cost function, it holds that  $\gamma = -\frac{1}{Q_m} \log(\frac{\epsilon}{\bar{c}})$ . This implies that a rise in  $\bar{c}$  has also the property of reducing the learning speed parameter. As shown in the table, when  $\bar{c}$  increases, also the initial trigger value  $X_0^*$  rises, that is the investment is delayed because the higher initial marginal cost must be compensated by a higher level of  $X$ . As in the previous cases, the response to a higher investment trigger value is an increased intensity of the initial investment  $\bar{K}$ .

It is worth emphasizing the striking consistency of these results. A rise in  $w$ ,  $\sigma$  and  $\bar{c}$ , and a decrease in  $\alpha$ , represent "bad news" for the firm, and the optimal reaction to these events is exactly the same: the investment is delayed but its intensity is increased when it occurs. This prediction follows the same lines of Bar-Ilan and Strange (1999). In a model in which the firm has the possibility of choosing the timing and the intensity of the investment, they show that, when the investment is lumpy, factors that delay the investment raise its intensity. The difference with our approach is that their analysis is based on a model that "by construction" investigates the properties of lumpy investments, while we analyze incremental (marginal) investment

$\rho$	0.08	0.09	0.1	0.12	0.16	0.2	0.24	0.28
$X_0^*(\sigma = 0.2)$	56.19	56.49	56.81	57.5	59.05	60.75	62.54	64.39
$\bar{K}(\sigma = 0.2)$	3.1	3.1	3	2.9	2.6	2.4	2.4	2.3
$X_0^*(\sigma = 0.3)$	70.83	70.39	70.08	69.78	69.98	70.80	71.97	73.37
$\bar{K}(\sigma = 0.3)$	6.7	6	5.9	5.2	4.5	3.8	3.1	3.4

Table 7: The effects of the discount factor on the initial investment threshold and size of the jump for different levels of volatility

decisions and lumpiness is an endogenously generated feature deriving from the effects of the learning curve.

Bar-Ilan and Strange (1999) explain their findings with the fact that the intensity of the investment is determined by the marginal productivity of capital. If the investment trigger is higher, for that level of selling price the marginal productivity of capital is larger and a larger level of initial capital is justified. In addition to this explanation, in our model also another effect is at work. When  $K = 0$ , if the economic condition worsens, the option to invest and the option to produce, although lower in absolute value, are both relatively larger with respect to the value of the additional unit. However, for positive levels of productive capacity, only the option value to invest (relatively) increases because the option to produce has been already exercised once and for all when the initial investment is made. For this reason, the rise in investment trigger for  $K = 0$  is more than proportional than the increments of  $X^*$  for  $K > 0$ . It follows that the conditions that determine the size of the initial investment are met for a higher level of capital  $\bar{K}$ .

An additional possible explanation for the increase in the size of the initial investment is based on a learning effect. Recall that a larger productive capacity guarantees a faster learning. Thus, the firm might be induced to compensate with a larger  $K$  the deterioration of the economic environment (that caused the delay of the investment) with a quicker reduction of the marginal cost level. This compensating effect will be clarified in the remaining part of the section where we investigate the effects of  $\rho$  and  $\gamma$  on the initial investment.

The effects of changes in the discount factor  $\rho$  and learning speed parameter  $\gamma$  depend on the interactions with other parameters values and requires a more elaborate discussion. In Bar-Ilan and Strange (1999), an increase in the discount factor  $\rho$  reduces the size of the investment while the effect on the investment trigger is ambiguous and depends on the choice of the other parameters values. This ambiguity is present also in our model. As shown in Table 7, for a volatility level of  $\sigma = 0.2$ , the trigger value  $X_0^*$  is monotonically increasing while  $\bar{K}$  is decreasing. But when the volatility rises to 0.3, the effect on  $X_0^*$  is ambiguous while  $\bar{K}$  is generally decreasing even if it rises in the step from  $\rho = 0.24$  to  $\rho = 0.28$ .

The effects of  $\gamma$  exhibit also ambiguities. Recall that in our model, in order to ensure continuity of the marginal cost function, the learning speed parameter is

$\gamma$	0.005	0.007	0.009	0.014	0.028	0.069	0.14	0.28
$X_0^*(\rho = 0.1)$	56.81	56.81	56.81	56.81	56.81	56.81	56.81	56.81
$\bar{K}(\rho = 0.1)$	2.9	2.9	2.9	3	4.2	6	6.7	7.1
$X_0^*(\rho = 0.24)$	62.54	62.54	62.54	62.54	62.54	62.54	62.54	62.54
$\bar{K}(\rho = 0.24)$	2.8	2.6	2.4	2.4	2.7	4.4	5.5	6.2

Table 8: The effects of the learning speed parameter on the initial investment threshold and size of the jump for different value of the discount factor

$\rho$	0.08	0.1	0.12	0.14	0.16	0.2	0.24	0.28
$X_0^*(\gamma = 0.014)$	56.19	56.81	57.5	58.25	59.05	60.75	62.54	64.39
$\bar{K}(\gamma = 0.014)$	3.1	3	2.9	2.8	2.6	2.4	2.4	2.3
$X_0^*(\gamma = 0.005)$	56.19	56.81	57.5	58.25	59.05	60.75	62.54	64.39
$\bar{K}(\gamma = 0.005)$	2.9	2.9	2.8	2.8	2.7	2.8	2.8	2.8

Table 9: The effects of the discount factor on the initial investment threshold and size of the jump for different value of the learning speed parameter

defined as  $\gamma = -\frac{1}{Qm} \log(\frac{c}{\epsilon})$ . For our analysis, we modify  $\gamma$  by changing the boundary level  $Qm$ . Note that the initial trigger value  $X_0^*$ , given by equation (25), doesn't depend on  $Qm$  and, consequently, remains constant for different levels of  $\gamma$ . *A priori*, a *decrease* in  $\gamma$  may trigger two contrasting effects on the initial investment  $\bar{K}$ . A "holding effect" that pushes the firm to reduce the size of the initial investment and a "pushing effect" that induces the firm to invest more. The holding effect is due to the fact that if  $\gamma$  decreases, the learning process is slower and each additional unit is less valuable so that the firm is less willing to invest. But, on the other hand, if  $\gamma$  is already small, the learning process is very slow and this raises the incentive to increase the productive capacity. The total change of  $\bar{K}$  in response to a modification in the parameter  $\gamma$  depends on which of these two contrasting effects prevails.

The results of the simulation are shown in Table 8. For a level of the discount factor  $\rho = 0.1$ , as the learning speed parameter  $\gamma$  *decreases*,  $\bar{K}$  is initially decreasing and then remains stable. Pushing and holding effects compensate each other for low values of  $\gamma$ , while for high values the holding effect is preponderant. If  $\rho = 0.24$ ,  $\bar{K}$  is initially decreasing and then becomes increasing. When the learning speed parameter  $\gamma$  is very low, the pushing effect is predominant because the firm is willing to compensate the small values of  $\gamma$  with a larger productive capacity. When  $\gamma$  is already sufficiently high the holding effect prevails.

Table 9 is instructive to analyze pushing and holding effects from a slightly different perspective. The table shows the effects of changes in the discount factor  $\rho$  for  $\gamma = 0.014$  and  $\gamma = 0.005$ . For  $\gamma = 0.014$ , the results are the same as the upper part of Table 7, thus the trigger value  $X_0^*$  is monotonically increasing while  $\bar{K}$  is decreasing. For  $\gamma = 0.005$  the size of the initial investment  $\bar{K}$  is initially slightly decreasing and

then slightly increasing<sup>8</sup>. When  $\gamma$  is very small, an increase in the discount factor initially induces a decline in the size of the initial lumpy investment because the additional unit of capital is less valuable. But when the firm discounts a lot future profits (higher  $\rho$ ) the initial investment will be larger. The underlying reason is that the firm wants to compensate the higher discounting with a faster learning.

If we read Table 9 column by column, that is keeping constant the discount factor and varying the learning speed parameter, we see that when switching from  $\gamma = 0.014$  to  $\gamma = 0.005$ , the size of the initial jump is decreasing for low values of  $\rho$  and increasing for high values of  $\rho$ . For instance, when  $\rho = 0.8$ ,  $\bar{K}$  decreases from 3.1 to 2.9, that is the holding effect prevails. When  $\rho = 0.28$ ,  $\bar{K}$  increases from 2.3 to 2.8, that is the pushing effect prevails. Pushing and holding effect exactly offset each other for  $\rho = 0.14$  when the initial size of the jump is  $\bar{K} = 2.8$  for both  $\gamma = 0.014$  and  $\gamma = 0.005$ .

To summarize and further clarify the combined effect of  $\gamma$  and  $\rho$ , the economic intuition goes as follows. A decrease in the learning speed parameter and an increase in the discount factor make the additional unit of capital *less* valuable and, taking into account that the investment threshold remains constant, this causes a *decline* in the size of the initial investment. But if  $\gamma$  is already low and  $\rho$  is already high, a further deterioration of the economic conditions, that is an increase in  $\rho$  or a decrease in  $\gamma$ , triggers an opposite reaction: the size of the initial jump will result larger. This happens because the firm wants to counterbalance the combined effect of slow learning and high discounting by increasing the productive capacity, and thus decreasing the time needed to reach  $\underline{c}$ . In other words, for high  $\rho$  and low  $\gamma$ , the benefits of learning are already small because the learning process is slow and the stream of profits is heavily discounted. For this reason, after a further increase in  $\rho$  or a decrease in  $\gamma$ , the firm is more willing to invest, even if the value of the additional unit is lower, because its priority is to accelerate the (inefficiently) slow-moving learning process.

## 5 Conclusion

In this article we investigated the properties of the optimal investment for a firm facing uncertainty, irreversibility and learning curve. The main result of our model can be summarized as follows. In a context of decreasing returns of capital and incremental investment, the presence of the learning curve endogenously generates a region of increasing returns and gives rise to jumps for the optimal investment choice.

When the cost function is characterized by a learning curve, an additional unit of capital, raising the per period rate of production, has the property of increasing the speed of learning, allowing the firm to quickly reach lower levels of marginal costs. This implies two main consequences. First, when the level of installed capacity is

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<sup>8</sup>Recall that  $\gamma$  doesn't influence  $X_0^*$ , thus the threshold values are the same in the upper and lower part of the table

positive but very small, despite the uncertain environment, the option of waiting is worthless because the learning possibilities remain unexploited. Hence, the incentive to anticipate the investment due to the learning curve offsets the value of waiting due to uncertainty, and the optimal investment strategy reduces approximately to the traditional NVP rule. Second, for an initial region, capital exhibit increasing marginal returns. Indeed, the first units of capital are not the most valuable ones because the highest marginal productivity is achieved for a larger level of productive capacity, such that the learning process is sufficiently fast. The presence of an increasing returns region determines jumps for the optimal investment behavior. Specifically, the initial investment will be lumpy and the firm will invest marginally in the subsequent units of capital.

Lumpiness in the investment decision has non-trivial implications for what concerns comparative static results: factors that delay the investment increase its intensity when it eventually occurs. It follows, for example, that when sectors are affected by both the learning curve and high volatility, firms will enter the market with a large productive capacity.

In our analysis we assume impossibility of suspension of the production and irreversibility of the capital choice. The model can be extended allowing for operational flexibility (with or without stopping costs) and/or (partial) reversibility. The economic intuition suggests that (partial) reversibility should increase the intensity of the initial lumpy investment. If the capital choice is reversible the firm will initial install a larger amount of capital in order to quickly exploit the learning effect and will re-sell the excess capacity in case of downturns. The effect of operational flexibility is *a priori* more difficult to pin down. On one hand the firm might be induced to increase the size of the initial investment in order to rapidly move down the learning curve. Indeed, should the output price fall in the future, the firm will have the possibility of reducing the per period rate of production. On the other hand, it must be taken into account that the initial jump in the investment behavior is a consequence of the endogenously generated region of increasing returns. In case of operational flexibility, the total value of each installed unit of capital is given by the expected value of the discounted stream of profits plus the value of the opportunity of temporary shutting down. If this last component of the total value is larger for the first units of capacity, the region of increasing returns will result smaller, and smaller will be also the size of the initial jump. In the extreme scenario, the initial increasing returns region could even disappear and the firm will always invest marginally. In that case, the optimal investment behavior will not exhibit jumps.

Beside assuming impossibility of suspension of the production and irreversibility of the capital choice, in our model we abstract from competition. The existing literature on the effects of the learning curve on strategic interactions mainly focused on price and quantity competition<sup>9</sup>. Further research can build up on our frame-

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<sup>9</sup>For references about the strategic implications of the learning curve hypothesis see Cabral and Riordan (1994).

work to investigate the implications of the learning curve hypothesis for the optimal investment behavior in a competitive market.

## 6 Appendix

### 6.1 Derivation of equation (9)

Here we derive equation (9) for the value the investment option  $\Delta F(X; Q; K)$ .

The Bellman Equation for the option to invest is given by

$$\Delta F(X; Q; K) = e^{-\rho dt} E_t[\Delta F(X + dX; Q + dQ; K)] \quad (\text{A.1})$$

Expanding the right hand side by Ito's Lemma yields

$$(1 - \rho dt) E_t[(\Delta F + \Delta F_Q \frac{dQ}{dt} dt + \alpha X \Delta F_X dt + \sigma X \Delta F_X dz + \frac{1}{2} \sigma^2 X \Delta F_{XX}^2 dt)] \quad (\text{A.2})$$

Omitting terms that go to zero faster than  $dt$  as  $dt \rightarrow 0$ , and recalling that  $\frac{dQ}{dt} = K$  and  $E_t[dz] = 0$ , we get

$$\Delta F + (\frac{1}{2} \sigma^2 X^2 \Delta F_{XX} + \alpha X \Delta F_X + K \Delta F_Q - \rho \Delta F) dt. \quad (\text{A.3})$$

Substituting in (A.1) and simplifying, we obtain

$$\frac{1}{2} \sigma^2 X^2 \Delta F_{XX} + \alpha X \Delta F_X + K \Delta F_Q - \rho \Delta F = 0. \quad (\text{A.4})$$

### 6.2 Numerical method

In order to solve equation (9) we employ the finite difference method. We approximate the derivatives with respect the continuous variables  $X$  and  $Q$  by finite different quotients and then apply these approximations to discretize the partial differential equation (9). The numerical method starts from the boundary level of cumulative production  $Q_m$  and proceeds backwards to the initial value  $Q = 0$ . In each step of the simulation we define the investment and non investment regions by finding the investment threshold  $X^*$ .

The approximation for the derivatives with respect  $X$  and  $Q$  is obtained by Taylor expansion. Given a smooth function  $f$  we can write

$$f(x + \Delta x) = f(x) + f'(x) \Delta x + \frac{1}{2} f''(x) (\Delta x)^2 + \frac{1}{6} f'''(x) (\Delta x)^3 \dots \quad (\text{A.5})$$

Substituting  $\Delta x$  with  $-\Delta x$ , we can also write

$$f(x - \Delta x) = f(x) - f'(x) \Delta x + \frac{1}{2} f''(x) (\Delta x)^2 - \frac{1}{6} f'''(x) (\Delta x)^3 \dots \quad (\text{A.6})$$

From (A.5), we obtain an approximation of the first derivative as

$$f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} - O(\Delta x). \quad (\text{A.7})$$

A more accurate approximation is obtained by subtracting (A.6) and (A.7) from each other:

$$f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} - O(\Delta x)^2, \quad (\text{A.8})$$

where the error term is of order  $(\Delta x)^2$  rather than  $\Delta x$  as in (A.7).

The approximation for the second derivative can be obtained by adding (A.5) and (A.6) and is given by

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{2\Delta x} - O(\Delta x)^2. \quad (\text{A.9})$$

Using the above approximations, the PDE (9) can be rewritten as

$$-K\Delta F_Q = \frac{1}{2}\sigma^2 X^2 \Delta F_{XX} + \alpha X \Delta F_X - \rho \Delta F, \quad (\text{A.10})$$

where the right hand side contains only derivatives with respect to the stochastic variable  $X$ , and the left hand side the derivative with respect  $Q$ . The level of installed capacity  $K$  is taken as given. The approximation of (9) can be split in two parts: first the discretization with respect the variable  $X$ , second the discretization with respect the variable  $Q$ .

We consider now the discretization of the right hand side. For each value of the cumulative production  $Q$ , the option to invest  $\Delta F(X; Q; K)$  is a function of the variable  $X$ . For computational purposes we choose a subset in  $\mathbb{R}^m$  defining a grid of values and indicate with  $M$  the number of points in the grid. In our numerical method the stochastic variable  $X$  assumes values from 1 to 100 and  $M = 300$ , so that the size of the step between two adjacent grid points is  $\Delta X = 0.33$ . Given the set of grid values, the option to invest  $\Delta F(X; Q; K)$  can be expressed as a vector  $\Delta Z(Q; K)$  of dimension 300.

Using the finite different method the right hand side of (A.10) can be expressed as an operation in  $\Delta Z(Q; K)$ , and given that this operation is linear it can be represented by a matrix. The approximations (A.8) and (A.9) lead to matrix representations of the first and second derivatives as indicated by  $D_1$  and  $D_2$ , where it holds that  $X_1 < X_2 < \dots < X_N$ . At the boundaries, i.e. top and bottom rows of the matrices  $D_1$  and  $D_2$ , (A.8) and (A.9) cannot be applied, thus we approximate the first derivative using (A.7) and the second derivative with zero, assuming that the option  $\Delta F(X; Q; K)$  is

a linear function of  $X$ .

$$D_1 = \frac{1}{\Delta X} \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 & -1 & 1 \end{pmatrix}$$

$$D_2 = \frac{1}{(\Delta X)^2} \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \end{pmatrix}$$

The complete operator of the right hand side of (A.10) is given by

$$D = 0.5\sigma^2 \mathbf{X}^2 D_2 + \alpha \mathbf{X} D_1 - \rho \mathbf{I}, \quad (\text{A.11})$$

where  $\mathbf{X}$  is a  $M \times M$  diagonal matrix with the chosen grid points of the stochastic variable  $X$  on the main diagonal, and  $\mathbf{I}$  is the  $M \times M$  identity matrix.

In order to solve the partial differential equation (9) we have also to discretize with respect to the cumulative production  $Q$ . After the discretization with respect to  $X$ , we are left with the ordinary differential equation

$$-K \frac{\partial \Delta Z(Q; K)}{\partial Q} = D \Delta Z(Q; K). \quad (\text{A.12})$$

This equation must be solved backwards, starting from the value at final period  $Z(Qm; K)$ , where  $Q = Qm$ , to the initial period's value  $Z(0; K)$ . Introducing the variable  $\Delta W(Q; K) = \Delta Z(Qm - Q; K)$  we can solve the differential equation forward. In terms of the new variable the equation (A.12) becomes

$$K \frac{\partial \Delta W(Q; K)}{\partial Q} = D \Delta W(Q; K), \quad (\text{A.13})$$

where  $W(Qm, K)$  is given and we must compute  $W(0; K)$ . Approximating the first derivative of the left hand side yields

$$K \left( \frac{\Delta W(Q + dQ; K) - \Delta W(Q; K)}{\Delta Q} \right) = D \Delta W(Q + dQ; K), \quad (\text{A.14})$$

where  $dQ = \frac{Qm}{N}$  is the size of the step between two adjacent grid points of  $Q$  and  $N$  is the number of steps chosen for the discretization. In our simulation we used  $N = 500$ .

Rearranging equation (A.14) yields

$$\Delta W(Q + \Delta Q; K) = \left( I - \frac{\Delta Q D}{K} \right)^{-1} \Delta W(Q; K). \quad (\text{A.15})$$

We use this formulation to solve the partial differential equation (9).

The complete simulation process proceeds as follows. We start from the boundary level of cumulative production  $Q = Qm$  and go backwards to the initial level of cumulative production  $Q = 0$ . The boundary  $Q = Qm$  can be used as starting point for the simulation because for that level of cumulative production we have the analytical solutions derived in section 3.2: the investment threshold  $X^*$  is obtained by equation (18), above the threshold the firm invest and the net value of the installed unit  $\Delta \bar{V}(X; K) - w$  is given by equation (13) minus the sunk installation cost  $w$ , while below the threshold the firm doesn't invest and the option value  $\Delta F(X; K)$  is given by equation (15).

After the first step of the simulation we define the investment and non investment regions by identifying the threshold  $X^*$ . To find  $X^*$  we exploit the value matching condition given by equation (11). Because of the discretization process, condition (11) is never exactly satisfied, thus we choose as threshold value  $X^*$  the smallest  $X$  for which it holds  $\Delta V(X^*; Q; K) - \Delta F(X^*; Q; K) - w \geq 0$ . Below  $X^*$  the firm doesn't invest and the value of the additional unit is given by the result of the simulation. Above  $X^*$  the firm does invest and the value of the additional unit is given by the expected value of the discounted stream of profits  $\Delta V(X; Q; K)$ . Thus, we substitute the results of the simulation with the values obtained by equation (8) and we can proceed with the second step. Given that the firm has the opportunity to exercise the option to invest at each point of time (like an American option), i.e. for each value of  $Q$ , the above described procedure is repeated for every step of the simulation.

## References

- [1] Abel, A. B. and J.C. Eberly, 1996. "Optimal Investment with Costly Reversibility," *Review of Economic Studies*, Vol. 63, No. 4, pp 581-593.
- [2] Aguerrevere, F., 2003, "Equilibrium Investment Strategies and Output Price Behavior: A Real-Options Approach", *The Review of Financial Studies*, Vol. 16, No. 4, pp. 1239 – 1272.
- [3] Arrow, K.J., 1968, "Optimal Capital Policy with Irreversible Investment", in *Value, Capital and Growth*, Edinburg University Press.

- [4] Bar-Ilan, A., and W.C. Strange, 1999, "The Timing and Intensity of Investment", *Journal of Macroeconomics*, Vol.21, No. 1, pp. 57-77.
- [5] Cabral, L.M.B, and M.H. Riordan, 1994, "The Learning Curve, Market Dominance, and Predatory Pricing", *Econometrica*, Vol. 62, pp. 1115-1140.
- [6] Dixit, A., 1995, "Irreversible Investment with Uncertainty and Scale Economies", *Journal of Economics Dynamics and Control*, Vol. 19, pp. 327-350.
- [7] Dixit, A. and R.S. Pindyck, 1994, *Investment Under Uncertainty*, Princeton University Press, Princeton, NJ.
- [8] Majd, S. and R.S. Pindyck, 1989, "The Learning Curve and the Optimal Production Under Uncertainty", *The RAND Economic Journal*, Vol. 20, No. 3, pp. 331-343.
- [9] Majd, S. and R.S. Pindyck, 1987, "Time to Build, Option Value, and Investment Decisions", *Journal of Financial Economics*, Vol. 5, pp 7-27
- [10] Pindyck R.S., 1988, "Irreversible Investment, Capacity Choice and the Value of the Firm", *The American Economic Review*, Vol. 78, No. 5, pp. 969-985.